EQUILIBRIUM CRACK IN A THIN LAYER

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The plane and axially symmetric problems of an equilibrium crack in a thin layer, clamped between two smooth, rigid foundations, are considered. The crack is located symmetrically relative to the edges of the layer and is maintained in an open state by stresses applied to its surface.

Equations are obtained that define the crack in its extended state.

We note that the problem of a crack in a layer was considered earlier only for the thick layer [1 and 2]; therefore it was not necessary to make use of complex mathematical techniques. For the solution of the problem indicated for a thin layer the derivation set forth in the papers [3 and 4] and the mathematical technique of Wiener-Hopf [5] are used. The lengthwise dimensions of the crack are determined by the method applied in [6].

1. **Plane problem of an equilibrium crack in a thin layer.** In an elastic layer of thickness $2h$ clamped between two smooth rigid foundations, let a longitudinal crack arise and be maintained in open state by normal stresses $q(x)$ applied to its surface. The whole system of stresses is statically equivalent to zero. The crack is located symmetrically relative to the edges of the layer. It is required to determine the shape of the crack $\gamma(x)$ and its half-length a in relation to the value of stress $q(x)$.

By methods of the operational calculus the problem considered may be reduced to the determination of a function $\gamma(x)$ from the following integral equation (*).

$$\int_{-a}^{a} \gamma(\xi)\, M\left(\frac{\xi - x}{h}\right) d\xi = -\frac{\pi h^2}{\Delta} q(x), \quad |x| \leqslant a \qquad \left(\Delta = \frac{E}{2(1-\sigma^2)}\right) \tag{1.1}$$

Here values E and σ are the elastic constants of the layer,

$$M(t) = -\int_0^\infty uL(u)\cos(ut)\,du \left(t = \frac{\xi - x}{h}\right), \quad L(u) = \frac{\sinh 2u + 2u}{\cosh 2u - 1} \tag{1.2}$$

Let us find $K(t)$ satisfying Equation

$$-K_t'(t) = M(t) \tag{1.3}$$

To within a constant we obtain

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*) The kernel $M(t)$ is regarded in the sense of generalized function.

$$K(t)=\int_0^\infty L(u)\sin(ut)\,du \tag{1.4}$$

Substituting $M(t)$ from (1.2) into Equation (1.1) and integrating once by parts, we get

$$\int_{-a}^{a}\gamma'(\xi)\,K\left(\frac{\xi-x}{h}\right)d\xi=-\frac{\pi hq(x)}{\Delta} \tag{1.5}$$

We note that the nonintegrand vanishes evidently by virtue of the condition $\gamma(\pm a)=0$.

Following [3], we obtain an asymptotic solution to Equation (1.5) for small values of the parameter $\lambda=h/a$, as the combination

$$\gamma'(\xi)=\omega\left(\frac{a+\xi}{h}\right)-\omega\left(\frac{a-\xi}{h}\right) \tag{1.6}$$

of the solutions $\omega(t)$ of an integral equation of the Wiener-Hopf type

$$\int_0^\infty\omega(\tau)\,K(\tau-t)\,d\tau=-\frac{\pi\psi(t)}{\Delta}\qquad(0\leqslant t<\infty) \tag{1.7}$$

Here the function $\psi(t)=q(ht-a)$, and is analytically continued in the region $2/\lambda\leqslant t<\infty$.

The solution to Equation (1.7) may be obtained in closed form by the method of Wiener-Hopf. However, in order to obtain the solution suitable for practical use, it is necessary to approximate the function $L(u)$ to an appropriate expression carrying out an approximate factorization [5].

Let us approximate the function $L(u)$ as follows:

$$L(u)=\frac{u^2+2D}{u\sqrt{u^2+D^2}} \tag{1.8}$$

It is easily seen that the approximation (1.8) reflects correctly the behavior of function $L(u)$ of the form (1.2) at zero and at infinity. For value $D=0.64$ the error of approximation in (1.8) does not exceed 8% for all $0\leqslant u<\infty$.

Subsequently we restrict ourselves to consider the case where $q(x)=q$ ($q=\mathrm{const}$), i.e. the crack is stressed uniformly by internal pressure. In this case the solution of integral equation (1.7) with approximation (1.8) is given by the relation

$$\omega(t)=\frac{q}{\Delta\sqrt{2\pi}}\left\{\frac{e^{-Dt}}{\sqrt{t}}-2\left(\sqrt{2D}-D\right)^{1/2}e^{-\sqrt{2D}\,t}F\left[\left(\sqrt{2D}-D\right)^{1/2}\sqrt{t}\right]\right\}\left(F(z)=\int_0^z e^{u^2}du\right) \tag{1.9}$$

Tables of the function $F(z)$ are available in [7].

The expression for $\gamma(x)$ with regard to $\gamma(\pm a)=0$ and (1.6) has the form

$$\gamma(x)=\int_x^a\left[\omega\left(\frac{a-\xi}{h}\right)-\omega\left(\frac{a+\xi}{h}\right)\right]d\xi \tag{1.10}$$

Substituting $\omega(t)$ in the form given in (1.9) and integrating, finally we obtain

$$\gamma(x)=\frac{qh}{2\Delta}\Big\{\operatorname{erf}\varphi(x)+\operatorname{erf}\varphi(-x)-\operatorname{erf}\varphi(a)+\frac{2A}{\sqrt{\pi}}\left[\exp\left(-B\varphi^2(x)\right)F\left(A\varphi(x)\right)+\right.$$
$$\left.+\exp\left(-B\varphi^2(-x)\right)F\left(-A\sqrt{\varphi(-x)}\right)-\exp\left(-B\varphi^2(a)\right)F\left(A\varphi(a)\right)\right]\Big\} \tag{1.11}$$

$$\varphi(x)=\frac{\sqrt{D(a+x)}}{\sqrt{h}},\qquad A=\left(\frac{\sqrt{2}}{\sqrt{D}}-1\right)^{1/2},\qquad B=\left(\frac{2}{D}\right)^{1/2} \tag{1.12}$$

We note that $\gamma(x)$ in the form of (1.12) does not reflect the shape of the crack in a small neighborhood of its tips [6].

It is easily seen from Equation (1.12) that for very small values of the parameter $\lambda = h/a$, the function $\gamma(x)$ defining the shape of the crack tends to the degenerate value

$$\gamma_0(x) = qh / 2\Delta \tag{1.13}$$

Assuming the crack in equilibrium, we proceed now to determination of its half-length a, using the method given in [6].

The coefficient of intensity of normal stress $\sigma_y(x, 0) = p(x)$ at $x > a$ (without taking into account cohesive forces) we will define by Equations

$$N_0 = \lim_{x\to a} \sqrt{x-a}\, p(x), \qquad p(x) = \frac{\Delta}{\pi h}\int_{-a}^{a} \gamma'(\xi) K\left(\frac{\xi - x}{h}\right) d\xi \tag{1.14}$$

We represent functions $\gamma'(\xi)$ and $K(t)$ in the form

$$\gamma'(\xi) = -\frac{q}{\Delta\sqrt{2\pi}}\left(\frac{h}{a-\xi}\right)^{1/2} + f(\xi), \qquad K(t) = \frac{1}{t} + E(t) \tag{1.15}$$

We can show that

$$\lim_{x\to a} \sqrt{x-a}\int_{-a}^{a}\left[\gamma'(\xi) E\left(\frac{\xi - x}{h}\right) + \frac{hf(\xi)}{\xi - x}\right] d\xi \to 0 \tag{1.16}$$

Then

$$N_0 = -\frac{q\sqrt{h}}{\pi\sqrt{2\pi}}\lim_{x\to a}\sqrt{x-a}\int_{-a}^{a}\frac{d\xi}{\sqrt{a-\xi}\,(\xi - x)} \tag{1.17}$$

Computing the integral, finally we get

$$N_0 = \frac{q\sqrt{h}}{\sqrt{2\pi}} = \frac{P\sqrt{h}}{2a\sqrt{2\pi}} \qquad (P = 2qa) \tag{1.18}$$

Here P is the total stress acting on the face of the crack.

Now, following [6] we obtain the equation for determination of the half-length a of the equilibrium crack

$$N_0 = \frac{K}{\pi}, \quad \text{or} \quad a = \frac{P^2\pi\lambda}{8K^2} \tag{1.19}$$

Here K is the modulus of cohesion, introduced by Barenblatt [6]. From Equation (1.19) it is easily seen that for the decrease of relative thickness of layer λ the half-length a of the crack is reduced proportionally for fixed stress P.

On the basis of calculations which have been carried out, Equations (1.11) and (1.19) are reliable for the case $\lambda \leqslant 2$.

2. Axially symmetric problem of an equilibrium crack in a thin layer. In an elastic layer of thickness $2h$ clamped between two smooth, rigid boundaries, let a longitudinal crack, round in the plan view, arise and be maintained in an open state by normal stresses $q(r)$ applied to its surface. The whole system of stresses is statically equivalent to zero. The crack is located symmetrically with respect to the edges of the layer.

It is required to determine the shape of the crack $\gamma(r)$ and its radius a in relation to the value of stresses $q(r)$.

The equation of the region accupied by the crack, obviously is given by the relation

$$x^2 + y^2 \leqslant a^2 \tag{2.1}$$

Let us introduce new variables by Equations [4]

$$t = (a - x)/h, \qquad \tau = y/h \tag{2.2}$$

Substituting in (2.1) and setting $\lambda = h/a \to 0$, we obtain

$$t \geqslant 0 \tag{2.3}$$

This means that in the case of small relative thickness of layer λ the round crack expands asymptotically into a crack, occupying the half-plane $t \geqslant 0$. .

The solution of this problem is evidently given by the relation [4]

$$\gamma_*(t) = h \int_0^t \omega(\xi)\, d\xi \tag{2.4}$$

where $\omega(t)$ is the solution to the integral equation of Wiener-Hopf (1.7), in which $\psi(t) = q(a - ht)$, and the function $\psi(t)$ is analytically continued in the region $2/\lambda \leqslant t < \infty$.

Taking into account the axial symmetry of the problem and returning to the old variables, we obtain

$$\gamma(r) = \gamma_*\left(\frac{a - r}{h}\right) \tag{2.5}$$

The conditions for determining the radius a of the crack are also obtained from the solution of the problem of a crack in the form of a half-plane.

Let us consider, for example, the case $q(r) = q$. On the basis of the above statement, from Equation (1.11) we obtain at once

$$\gamma(r) = \frac{qh}{2\Delta}\left\{\operatorname{erf}\varphi(-r) + \frac{2A}{\sqrt{\pi}}\exp(-B\varphi^2(-r))\, F(A\,\varphi(-r))\right\} \tag{2.6}$$

Here φ , A and B are defined by (1.12).

The coefficient of intensity of normal stresses (1.18) is given by Equation

$$N_0 = \frac{q\sqrt{h}}{\sqrt{2\pi}} = \frac{P\sqrt{h}}{\pi a^2\sqrt{2\pi}} \qquad (P = q\pi a^2) \tag{2.7}$$

Here P is the total stress acting on the face of the crack. Then the condition for determination of radius a of an equilibrium crack is found in the form

$$a = \left(\frac{P\sqrt{\lambda}}{K\sqrt{2\pi}}\right)^{2/3} \tag{2.8}$$

From Equation (2.8) it is seen that as the relative thickness of layer λ is reduced, the radius a of the crack decreases considerably slower.

On the basis of calculatoons which have been carried out, Formulas (2.6) and (2.8) are reliable for the case $\lambda \leqslant 2$.

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